

MORTALITY IN STOCHASTIC POPULATION PROJECTIONS

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Abstract: This paper discusses a one-parameter model of the survival function intended for use with stochastic population projections. Section 1 introduces the model, a one-parameter version of the Brass logit life table system, its estimation, and illustrates its precision using Danish life tables. Section 2 references Swedish life tables and illustrates applicability of the model in the perspective of population projections. Section 3 illustrates the model as a means of simulating stochastic survivorship. Section 4 illustrates how the method embeds stochastic survivorship in projected populations. It is concluded that this could be a suitable survival model for use with stochastic population projections.

1.0 INTRODUCTION

1.1 Relational survival

Life table survival functions can be linearly related in the sense that for a chosen standard survival function l_x^S , radix one, another survival function l_x can be expressed as

$$\text{logit } l_x \approx \alpha + \beta \cdot \text{logit } l_x^S \quad (1.1)$$

where α and β are parameters and $\text{logit } p = \ln \frac{1-p}{p}$ with $0 < p < 1$ (see e.g., Brass, 1971, 1974 and 1975; Brass *et al.*, 1968; and Hill and Trussell, 1977). It follows from (1.1) that

$$l_x \approx 1 / (1 + e^{\beta \cdot ((1-l_x^S)/l_x^S) + \alpha}) \quad (1.2)$$

To fit the right-hand side of (1.1) to an observed or *a priori* given survival function

l_x , $\sum_{x=1}^{\infty} \left[\text{logit } l_x - \alpha - \beta \cdot \text{logit } l_x^S \right]^2$ is minimized with respect to α and β . Denoting

the estimated parameters by $\hat{\alpha}$ and $\hat{\beta}$, fitted survival is

$$\hat{l}_x(\hat{\alpha}, \hat{\beta}) = 1 / (1 + e^{\hat{\beta} \cdot ((1-l_x^S)/l_x^S) + \hat{\alpha}}) \quad (1.3)$$

This unweighted least-squares approach to estimating the parameters in (1.1) is usually adequate for practical work (for alternative estimation procedures, see e.g., Carrier and Goh, 1972). The significance of the parameters is that, relative to the chosen standard survival function, α is related to the level of mortality while β controls the relationship between childhood and adult mortality (see e.g., Brass, 1974 and Brass *et al.*, 1968). For (1.3) to provide an adequate approximation to an observed survival function l_x , l_x^S should have approximately the same age-pattern and level of mortality as l_x . In such cases, $\hat{\beta} \approx 1$ (see e.g., Brass, 1971).

In this paper, we adopt the one-parameter model ($\beta = 1$)

$$l_x(\alpha) = 1 / (1 + e^{\alpha \cdot (1-l_x^S)/l_x^S}) \quad (1.4)$$

To estimate α , the unweighted sum of squares

$$\sum_{x=1}^{104} \left[1_x - 1/(1 + e^{\alpha} (1 - 1_x^S)/1_x^S) \right]^2 \quad (1.5)$$

at ages between 1 and 104 is minimized with respect to α . The main purpose of the paper is to discuss the possibility of letting (1.4) serve as a survival model when making stochastic population projections. Before we discuss this possibility, we turn to an illustration of (1.4) using Danish life tables.

1.2 Illustrative application to Danish life tables

Table 1.1 shows estimates of α when using 1921 Danish male and female survival as standards when fitting (1.4) to Danish male and female survival for the period 1922-51. As an illustration, fig. 1.1 shows the result of fitting (1.4) to 1936 Danish male survival (using 1921 Danish male survival as a standard). In this experiment, $\hat{\alpha} = -0.097$ (table 1.1). The life expectancy of the standard is $e_o^S = 61.0$ years, for observed survival $e_o = 62.5$, and for the fitted (the estimated model) $e_o^f = 62.4$ years. This suggests that for a displacement in life expectancy of about one and a half years relative to the standard, (1.4) provides a close fit.

Table 1.1. Estimates of α for 1922-51 using 1921 Danish male and female survival as standards.

Year	α Males	α Females	Year	α Males	α Females
1921	0.000	0.000	1937	-0.118	-0.191
1922	0.075	0.083	1938	-0.196	-0.254
1923	0.042	0.024	1939	-0.248	-0.325
1924	0.042	0.027	1940	-0.285	-0.335
1925	-0.020	-0.018	1941	-0.267	-0.335
1926	0.008	-0.008	1942	-0.357	-0.416
1927	0.025	0.042	1943	-0.378	-0.435
1928	-0.003	-0.030	1944	-0.304	-0.354
1929	-0.016	-0.021	1945	-0.274	-0.339
1930	-0.028	-0.055	1946	-0.368	-0.405
1931	0.006	-0.028	1947	-0.444	-0.525
1932	-0.067	-0.068	1948	-0.548	-0.695
1933	-0.117	-0.130	1949	-0.550	-0.675
1934	-0.136	-0.187	1950	-0.564	-0.675
1935	-0.069	-0.098	1951	-0.609	-0.729
1936	-0.097	-0.144			

Fig. 1.2 shows similar graphs for Danish females. Here 1921 Danish female survival serves as a standard. For this standard $e_o^S = 62.5$, for observed 1936 female survival $e_o = 64.6$, and for the fitted $e_o^f = 64.4$ years.

Fig. 1.1. Observed and fitted survival for Danish males 1936 using 1921 Danish male survival as a standard.

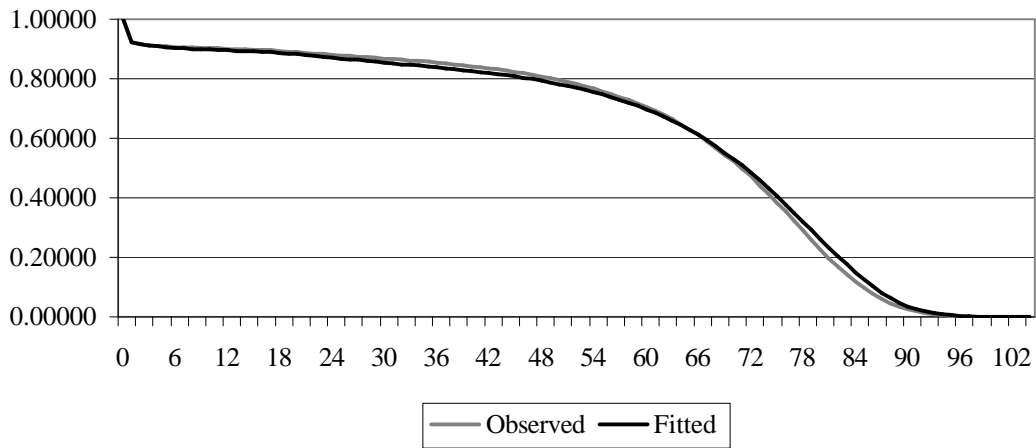
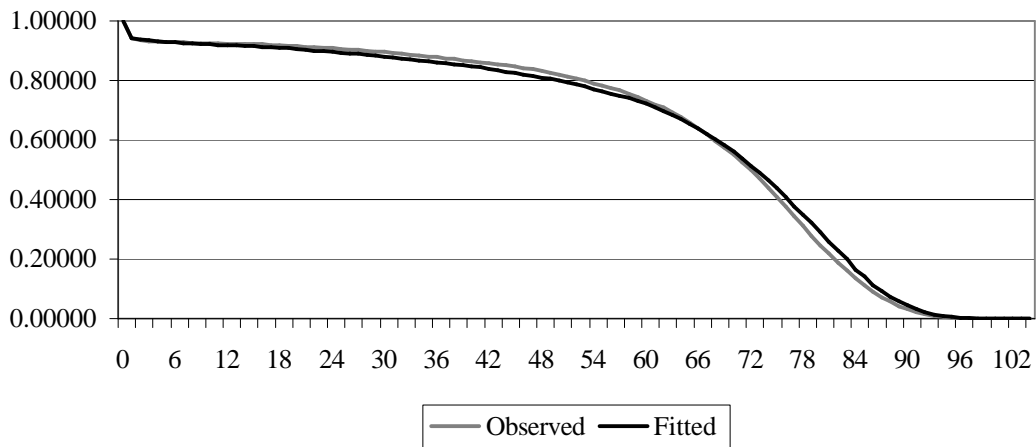


Fig. 1.2. Observed and fitted survival for Danish females 1936 using 1921 Danish female survival as a standard.



1.3 Modeling population projection probabilities

In life table notation, the projection probability is,

$$p(x) = L(x+1)/L(x) \quad (1.6)$$

where $L(x)$ are the expected person-years (exposure) in the chosen life table for an individual aged x and $p(x)$ is the survival probability for a life aged x to survive to age $x+1$. Because the survival function

$$s(x) = (1 - q_0) \dots (1 - q_{x-1})$$

is the product of terms $(1 - q_i)$ where q_i , $i = 0, \dots, x-1$, is the probability for a life aged i to die before reaching age $i+1$ (the life table mortality rate), different sets of life table mortality rates may, numerically speaking, lead to virtually indistinguishable survival functions. Since, moreover, $L(x)$ usually is approximated by

$$L(x) = [(s(x) + s(x+1))/2],$$

projection probabilities that only differ marginally may result from rather different underlying mortality rates (as long as they share approximately the same level of mor-

tality). For example, increasing $q_x = 0.1$ by 10 percent only changes $p_x = 1 - q_x$ from 0.90 to 0.89; a 1.1 percent decline. Indeed, raising all mortality rates in the Swedish 2000 male life table by 10 percent only lowers its life expectancy at birth from 77.4 to 76.4 years; a 1.3 percent change. Stated differently, what in its own right may appear to be a significant change in mortality rates may involve a mere trifling change in survivorship. To illustrate this in more detail, fig. 1.3 shows projection probabilities as determined from the observed life table for Danish males in 1936 as well as those modeled by means of (1.4) using 1921 Danish male survival as a standard. While obviously there are some differences between the observed and fitted survival probabilities, intuitively these seem more or less negligible. Fig. 1.4 shows the corresponding graphs for Danish females in 1936.

Table 1.2 shows observed and fitted projection probabilities corresponding to the survival in fig. 1.3. The error between observed and fitted projection probabilities is of the order of 0.1 percent. These results have been shown merely to provide the reader with an understanding of the functionality and the potential use of (1.4) in the context of making population projections. We now turn to an application of (1.4) to Swedish life tables and illustrative projections of the Swedish population.

Fig. 1.3. Observed and fitted projection probabilities for Danish males in 1936 (using 1921 Danish male survival as a standard).

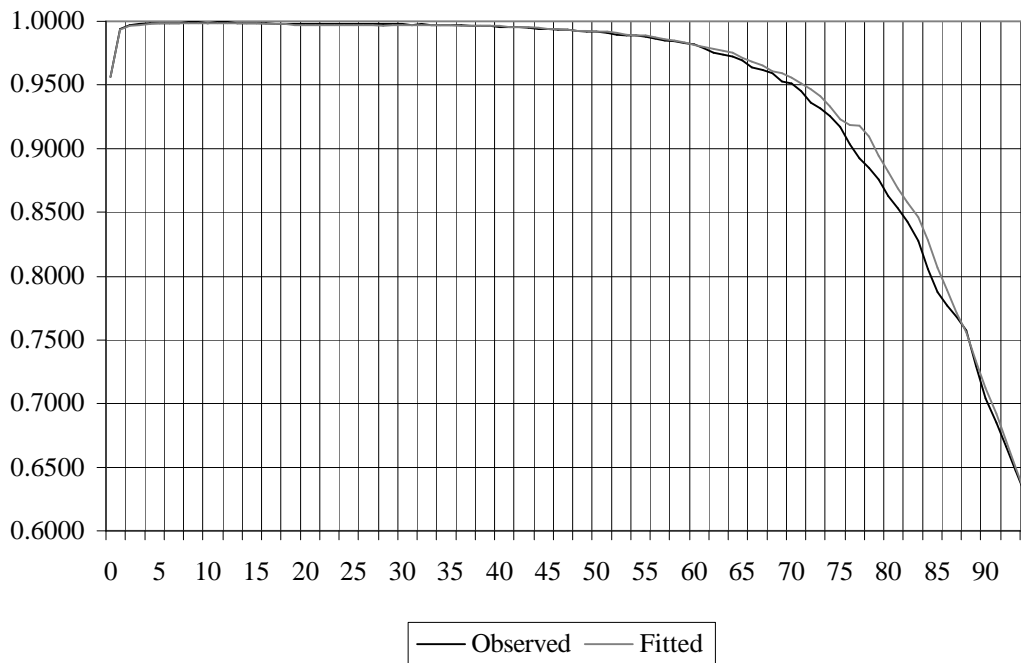
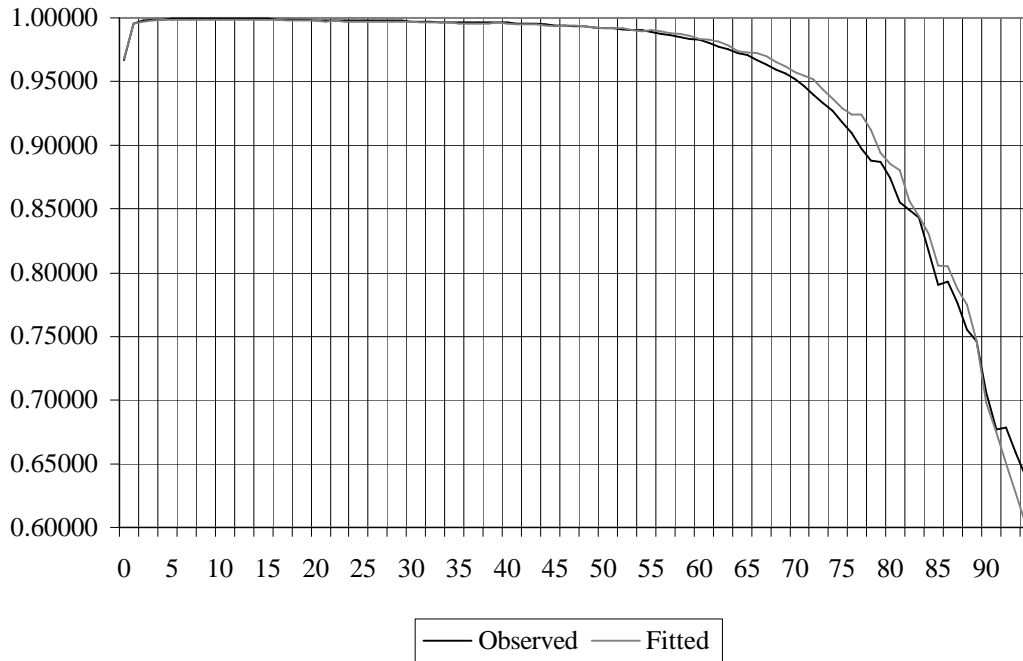


Table 1.2. Observed and fitted projection probabilities for Danish males in 1936 using 1921 Danish male survival as a standard.

Age	Projection probability		Percent error	Age	Projection probability		Percent error
	Observed	Fitted			Observed	Fitted	
0	0.956	0.956	0.04	48	0.993	0.992	0.04
1	0.994	0.993	0.08	49	0.992	0.993	-0.07
2	0.997	0.996	0.06	50	0.992	0.992	-0.05
3	0.998	0.997	0.04	51	0.991	0.991	-0.07
4	0.998	0.998	0.04	52	0.990	0.991	-0.12
5	0.999	0.998	0.04	53	0.989	0.989	-0.03
6	0.999	0.998	0.04	54	0.989	0.988	0.03
7	0.999	0.998	0.06	55	0.988	0.989	-0.06
8	0.999	0.998	0.07	56	0.986	0.987	-0.08
9	0.999	0.999	0.04	57	0.985	0.986	-0.06
10	0.999	0.999	0.02	58	0.984	0.985	-0.11
11	0.999	0.999	0.04	59	0.983	0.984	-0.09
12	0.999	0.999	0.04	60	0.982	0.982	0.01
13	0.999	0.999	0.03	61	0.979	0.980	-0.13
14	0.999	0.998	0.03	62	0.975	0.978	-0.35
15	0.999	0.998	0.03	63	0.974	0.977	-0.34
16	0.998	0.998	0.05	64	0.972	0.975	-0.30
17	0.998	0.998	0.01	65	0.969	0.972	-0.24
18	0.997	0.997	0.01	66	0.964	0.968	-0.44
19	0.998	0.997	0.07	67	0.962	0.965	-0.33
20	0.998	0.997	0.07	68	0.959	0.961	-0.16
21	0.997	0.997	0.06	69	0.953	0.959	-0.67
22	0.998	0.997	0.09	70	0.951	0.955	-0.49
23	0.998	0.997	0.09	71	0.945	0.951	-0.62
24	0.998	0.997	0.09	72	0.936	0.946	-1.11
25	0.998	0.997	0.08	73	0.931	0.941	-1.09
26	0.998	0.997	0.06	74	0.925	0.933	-0.78
27	0.998	0.997	0.11	75	0.917	0.923	-0.64
28	0.998	0.997	0.15	76	0.904	0.919	-1.67
29	0.998	0.997	0.12	77	0.892	0.918	-2.83
30	0.998	0.997	0.04	78	0.885	0.910	-2.79
31	0.997	0.997	0.01	79	0.875	0.895	-2.19
32	0.997	0.997	0.04	80	0.863	0.881	-2.18
33	0.997	0.997	0.01	81	0.853	0.869	-1.84
34	0.997	0.997	-0.04	82	0.843	0.857	-1.74
35	0.997	0.997	-0.01	83	0.827	0.846	-2.34
36	0.997	0.996	0.03	84	0.806	0.828	-2.78
37	0.996	0.996	0.02	85	0.788	0.807	-2.42
38	0.996	0.996	0.03	86	0.777	0.789	-1.52
39	0.996	0.996	-0.02	87	0.768	0.771	-0.33
40	0.995	0.996	-0.08	88	0.758	0.756	0.23
41	0.995	0.996	0.00	89	0.730	0.733	-0.44
42	0.995	0.995	0.01	90	0.704	0.713	-1.16
43	0.995	0.995	-0.06	91	0.685	0.692	-0.96
44	0.994	0.995	-0.05	92	0.666	0.671	-0.71
45	0.994	0.994	0.01	93	0.647	0.650	-0.41
46	0.993	0.994	-0.02	94	0.628	0.629	-0.14
47	0.993	0.993	0.04	95+	0.586	0.584	0.45

Fig. 1.4. Observed and fitted projection probabilities for Danish females in 1936 (using 1921 Danish female survival as a standard).



2.0 PROJECTIONS OF THE SWEDISH POPULATION

2.1 Projecting the 1980 population to the year 2000

Table 2.1 shows estimates of α for Swedish males and females 1981-2002 using 1980 Swedish male and female survival as standards. Fig. 2.1 shows the observed and estimated survival functions for Swedish males in the year 2000. The life expectancy for observed survival is $e_o = 77.4$ and for fitted $e_o^f = 77.0$ years. The life expectancy for the standard is $e_o^s = 72.8$ years. Although here the difference in life expectancy between the standard and the modeled survival functions is about 4.5 years, the fit remains close.

It is in place to note that when the difference in life expectancy between the standard and the survival to be modeled is small (a difference of some one or two years), the two curves can hardly be distinguished in a graph.

Fig. 2.2 shows projection probabilities for observed and modeled survival functions for Swedish males in the year 2000. Once again, it is apparent that the differences between the observed and modeled projection probabilities are marginal.

Notwithstanding the smallness of the differences, their computational significance can only be *gauged objectively* if the Swedish population is projected using partly observed, partly modeled probabilities.

Fig. 2.1. Observed and estimated survival functions for Swedish males in the year 2000 using 1980 Swedish male survival as a standard

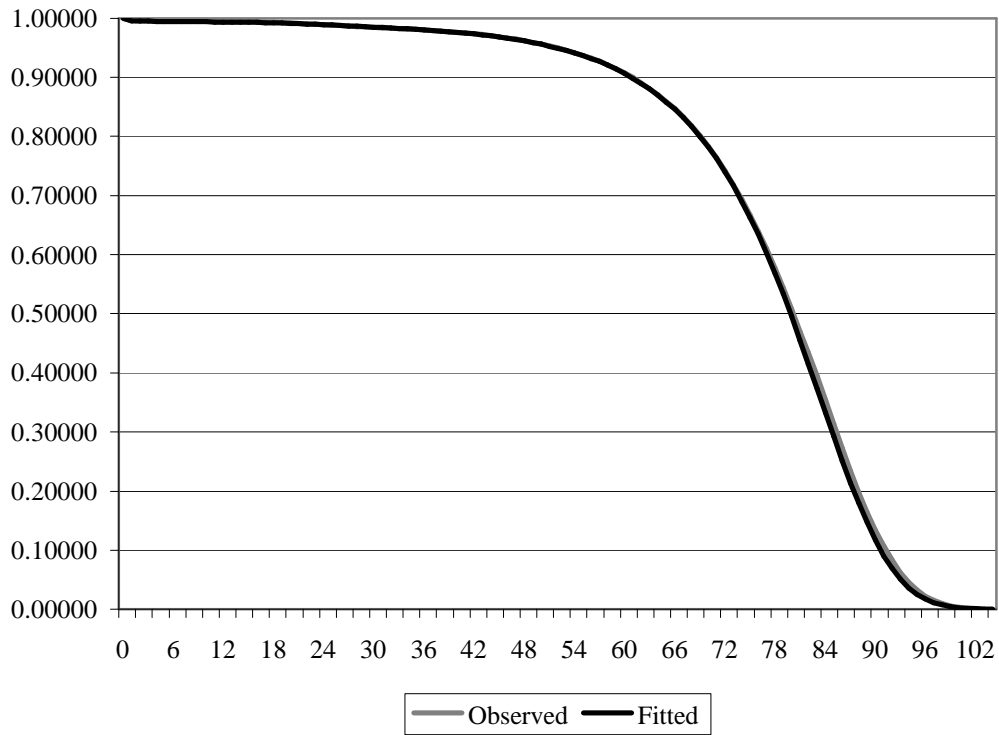


Table 2.1. Estimates of α for Swedish males and females 1981-2002 using 1980 Swedish male and female survival as standards.

Year	Males	Females	Year	Males	Females
1980	0.000	0.000	1992	-0.325	-0.282
1981	-0.029	-0.036	1993	-0.345	-0.281
1982	-0.069	-0.078	1994	-0.419	-0.374
1983	-0.100	-0.122	1995	-0.431	-0.375
1984	-0.128	-0.153	1996	-0.475	-0.389
1985	-0.119	-0.129	1997	-0.500	-0.432
1986	-0.147	-0.171	1998	-0.531	-0.451
1987	-0.174	-0.191	1999	-0.560	-0.441
1988	-0.173	-0.162	2000	-0.560	-0.464
1989	-0.259	-0.259	2001	-0.639	-0.474
1990	-0.261	-0.233	2002	-0.660	-0.478
1991	-0.276	-0.258			

Fig. 2.2. Observed and fitted projection probabilities for Swedish males in 2000 using 1980 Swedish male survival as standard.

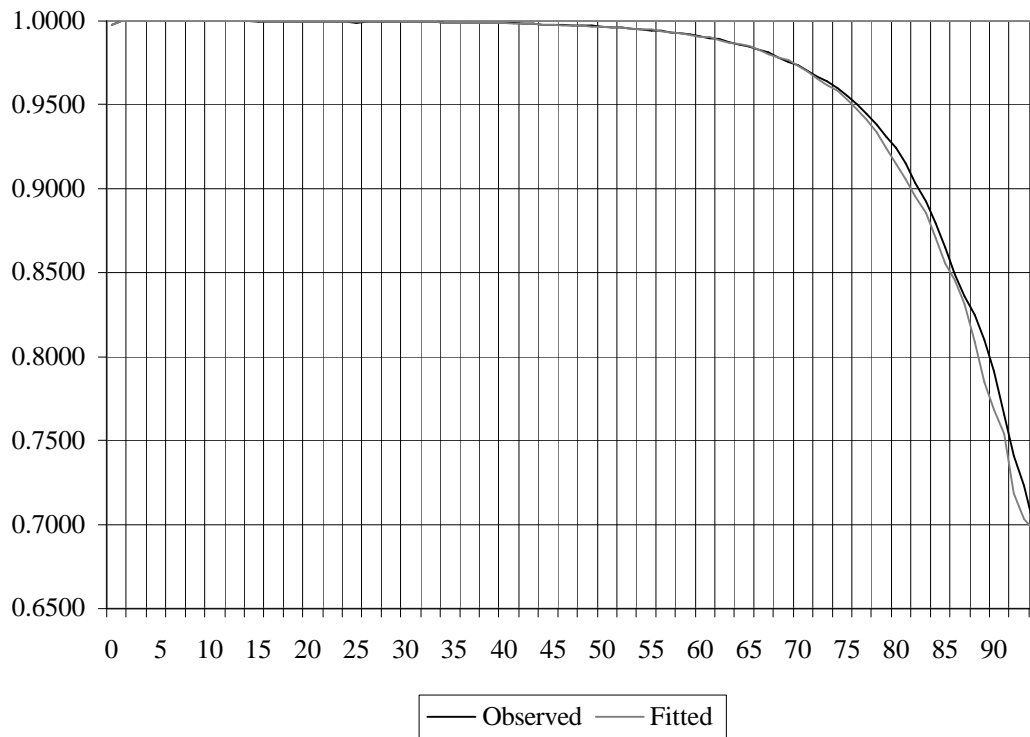


Table 2.2 shows the results of projecting the Swedish 1980 midyear population 30 years into the future using partly projection probabilities estimated directly from the 2000 survival functions for males and females, partly by using projection probabilities derived from the modeled 2000 survival functions for males and females. For example, modeled male survival is

$$\hat{l}_x^S = 1 / (1 + e^{-0.560} (1 - l_x^S) / l_x^S)$$

(see table 2.1) where l_x^S denotes 1980 Swedish male survival.

Table 2.2. Projections of the Swedish 1980 midyear population 30 years into the future assuming a total fertility rate of 2.0, zero net-migration and projection probabilities corresponding to the observed and the modeled 2000 life table.

Year	Projection 1			Projection 2			Percent error		
	Both sexes	Males	Females	Both sexes	Males	Females	Both sexes	Males	Females
0	8,310,335	4,117,603	4,192,732	8,310,335	4,117,603	4,192,732	0.00	0.00	0.00
1	8,361,945	4,143,226	4,218,719	8,359,793	4,141,653	4,218,140	0.03	0.04	0.01
2	8,410,154	4,167,237	4,242,917	8,405,992	4,164,182	4,241,810	0.05	0.07	0.03
3	8,454,838	4,189,584	4,265,254	8,448,829	4,185,149	4,263,680	0.07	0.11	0.04
4	8,495,973	4,210,247	4,285,726	8,488,266	4,204,519	4,283,747	0.09	0.14	0.05
5	8,533,625	4,229,259	4,304,366	8,524,369	4,222,329	4,302,040	0.11	0.16	0.05
6	8,567,901	4,246,674	4,321,227	8,557,242	4,238,631	4,318,611	0.12	0.19	0.06
7	8,599,005	4,262,596	4,336,409	8,587,082	4,253,523	4,333,559	0.14	0.21	0.07
8	8,627,170	4,277,137	4,350,033	8,614,108	4,267,105	4,347,003	0.15	0.23	0.07
9	8,652,651	4,290,423	4,362,228	8,638,575	4,279,497	4,359,078	0.16	0.25	0.07
10	8,675,708	4,302,582	4,373,126	8,660,730	4,290,836	4,369,894	0.17	0.27	0.07
11	8,696,621	4,313,753	4,382,868	8,680,862	4,301,259	4,379,603	0.18	0.29	0.07
12	8,715,634	4,324,054	4,391,580	8,699,209	4,310,876	4,388,333	0.19	0.30	0.07
13	8,732,981	4,333,601	4,399,380	8,716,004	4,319,806	4,396,198	0.19	0.32	0.07
14	8,748,867	4,342,494	4,406,373	8,731,439	4,328,142	4,403,297	0.20	0.33	0.07
15	8,763,396	4,350,778	4,412,618	8,745,615	4,335,930	4,409,685	0.20	0.34	0.07
16	8,776,605	4,358,460	4,418,145	8,758,551	4,343,177	4,415,374	0.21	0.35	0.06
17	8,788,436	4,365,515	4,422,921	8,770,202	4,349,852	4,420,350	0.21	0.36	0.06
18	8,798,865	4,371,910	4,426,955	8,780,511	4,355,907	4,424,604	0.21	0.37	0.05
19	8,807,884	4,377,631	4,430,253	8,789,471	4,361,327	4,428,144	0.21	0.37	0.05
20	8,815,439	4,382,644	4,432,795	8,797,052	4,366,078	4,430,974	0.21	0.38	0.04
21	8,821,590	4,386,979	4,434,611	8,803,286	4,370,177	4,433,109	0.21	0.38	0.03
22	8,826,448	4,390,678	4,435,770	8,808,247	4,373,654	4,434,593	0.21	0.39	0.03
23	8,830,104	4,393,761	4,436,343	8,812,058	4,376,557	4,435,501	0.20	0.39	0.02
24	8,832,724	4,396,297	4,436,427	8,814,891	4,378,967	4,435,924	0.20	0.39	0.01
25	8,834,481	4,398,371	4,436,110	8,816,913	4,380,960	4,435,953	0.20	0.40	0.00
26	8,835,528	4,400,046	4,435,482	8,818,248	4,382,582	4,435,666	0.20	0.40	0.00
27	8,836,099	4,401,420	4,434,679	8,819,153	4,383,939	4,435,214	0.19	0.40	-0.01
28	8,836,476	4,402,627	4,433,849	8,819,925	4,385,175	4,434,750	0.19	0.40	-0.02
29	8,836,913	4,403,798	4,433,115	8,820,760	4,386,382	4,434,378	0.18	0.40	-0.03
30	8,837,666	4,405,051	4,432,615	8,821,905	4,387,664	4,434,241	0.18	0.39	-0.04

The population is projected with the assumption that the total fertility rate is constant at TFR = 2.0, zero net-migration and projection probabilities corresponding to the observed 2000 life table (Projection 1) and the modeled 2000 life table (Projection 2).

Projection 1 and Projection 2 confirm that differences between observed and modeled projection probabilities are of no consequence for the projected population sizes. The differences in percent between the two projections are shown in table 2.2. The two sets of projection probabilities also lead to similar age distributions. For example, in Projection 1 in the year 2010, the proportions aged 65+ are 17.3 and 20.8 percent for males and females and 17.0 and 20.8, respectively, in Projection 2. In the light of a population projection or forecast, these are insignificant differences.

3.0 STOCHASTIC VARIATION IN MORTALITY

3.1 Embedding natural variation in the projections

Figs. 3.1-3.4 show estimates of α for 1950-70 and 1980-2002 for males and females, respectively. In the case of 1950-70, 1950 survival is the standard. In the case of 1980-2002, it is 1980 survival that serves as standard.

It will be noted that the time-patterns are nearly linear for which reason the corresponding regression lines have been shown. The time-patterns in α for the Danish life tables (table 1.1) bring forth similar linear features. This is an advantageous feature of (1.4) since generally changes in life expectancy become linear changes in \bullet .

Fig. 3.1. Time progression for alpha, Swedish males 1970-1970

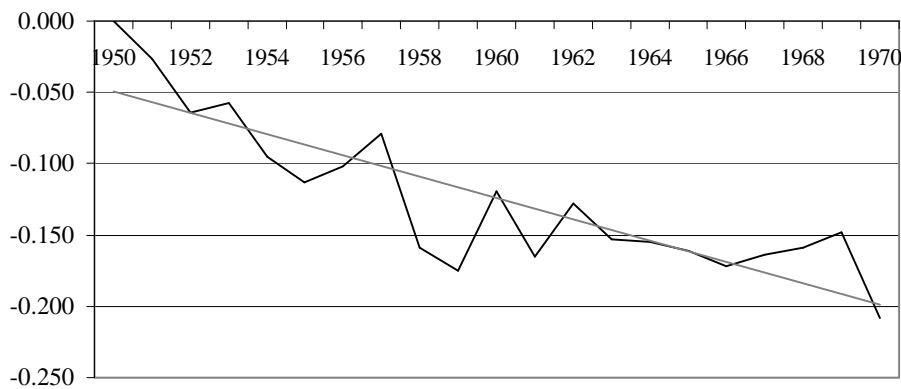


Fig. 3.2. Time progression for alpha, Swedish females 1950-70

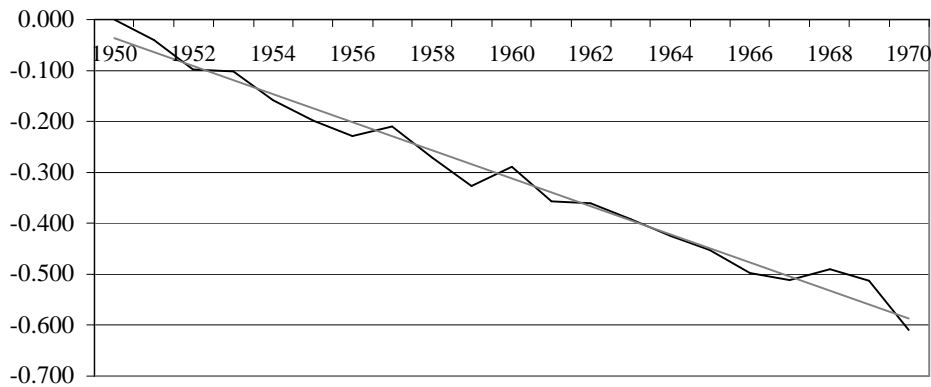


Fig. 3.3. Time progression for alpha, Swedish males 1980-2002

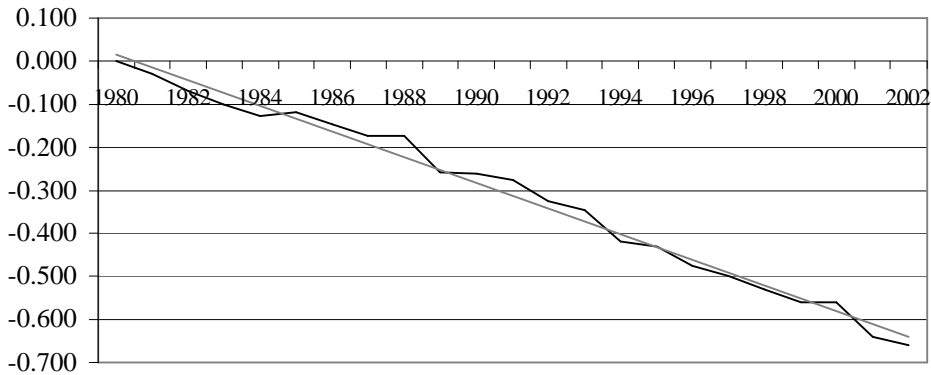
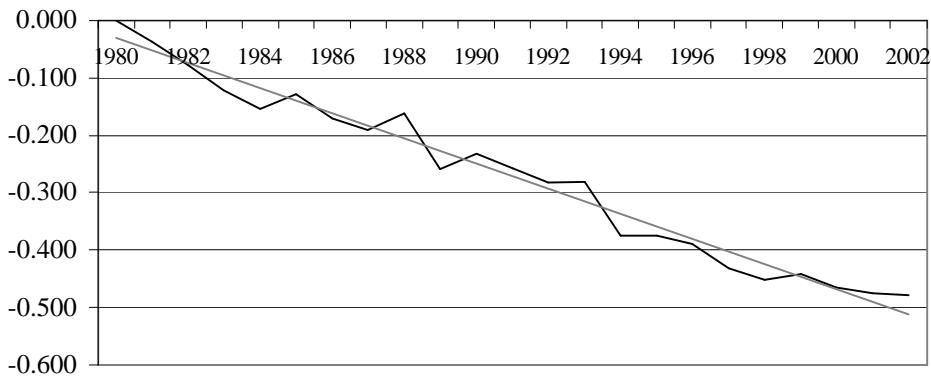


Fig. 3.4. Time progression for alpha, females 1980-2002



It seems reasonable, then, to model the time progression of α_t as

$$\alpha_t = h + kt + e_t,$$

that is, as a linear trend plus white noise. This means that residuals

$$d_t = \alpha_t - h - kt$$

should behave like white noise.

For the period 1950-70, we find for males

$$\alpha_t = -0.0492 - 0.0075t + e_t, \quad \sigma_e^m = 0.028 \quad (3.1)$$

and for females

$$\alpha_t = -0.0355 - 0.0275t + e_t, \quad \sigma_e^f = 0.025 \quad (3.2)$$

$t = 0, \dots, 20$.

For the period 1980-2002, we find for males

$$\bullet_t = 0.0162 - 0.03t + e_t, \bullet_e^m = 0.023 \quad (3.3)$$

and for females

$$\bullet_t = -0.031 - 0.022t + e_t, \bullet_e^f = 0.025 \quad (3.4)$$

$t = 0, \dots, 22$.

It can be shown that for (3.1) – (3.4) the residuals perform like white noise with the indicated standard deviations. It will be seen that for all four representations,

$$\bullet_e \approx 0.03.$$

For illustrative purposes, fig. 3.5 and fig. 3.6 show the virtually linear relationships between life expectancy and \bullet for Sweden 1980-2002.

Fig. 3.5. The relationship between α and the life expectancy at birth for Swedish males, 1980-2002.

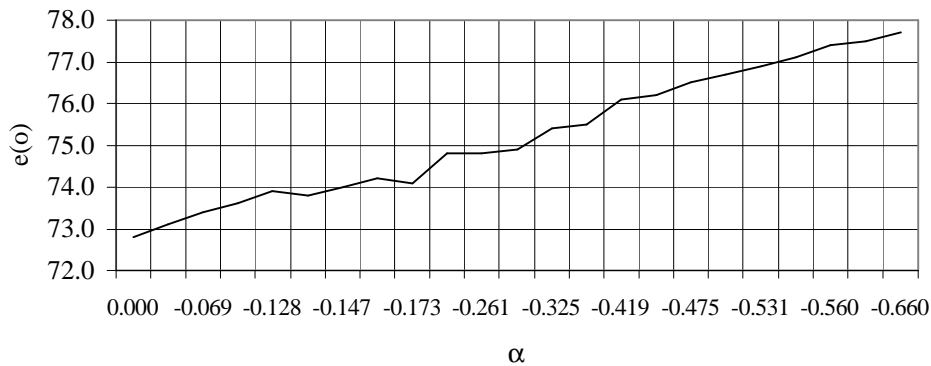


Fig. 3.6. The relationship between α and life expectancy at birth for Swedish females, 1980-2002

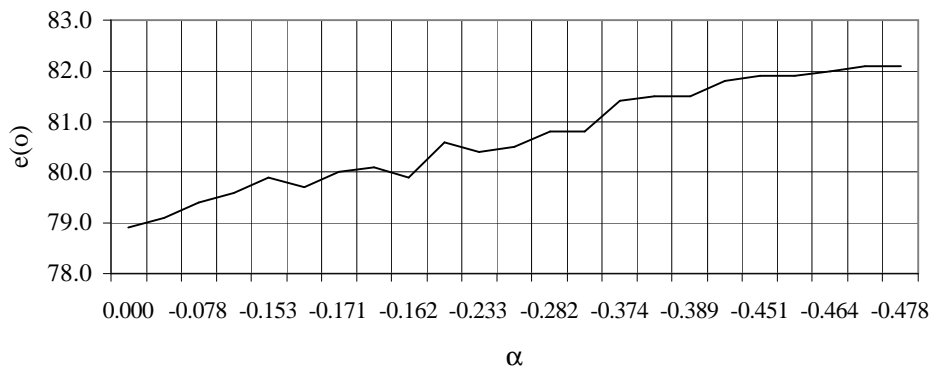


Table 3.1 Estimated coefficients for regression representations

$\hat{\bullet}(t) = \hat{k}t + \hat{h}$, and standard errors about regression lines (white noise).
Sweden 1950-70 and 1980-2002 and Denmark 1921-51.

Period	\hat{k}	\hat{h}	$\hat{\bullet}_e$	t
Sweden				
Males				
1950-70	-0.0075	-0.0492	0.028	$0 \leq t \leq 21$
1980-2002	-0.0298	0.0162	0.023	$0 \leq t \leq 22$
Females				
1950-70	-0.0275	-0.0355	0.025	$0 \leq t \leq 21$
1980-2002	-0.0219	-0.0305	0.025	$0 \leq t \leq 22$
Denmark				
1921-51				
Males	-0.0216	0.136	0.064	$0 \leq t \leq 30$
Females	-0.0255	0.148	0.075	$0 \leq t \leq 30$

$\hat{\bullet}_e$ is the standard deviation for white noise.

3.2 The variance of the life expectancy

The variance of the life expectancy can be interpreted in two different senses; (i) from the point of view of sampling from a super population or (ii) from the point of view of being a product from a realization of a stochastic process.

For single-year age groups, the asymptotic sampling variance of the estimated remaining life expectancy at age x , \hat{e}_x , is

$$\hat{\bullet}^2(\hat{e}_x) \approx \sum_{i=x}^{\infty} \hat{p}_{xi}^{-1} \left[0.5 + \hat{e}_{i+1} \right]^2 \frac{q_i^2 (1 - q_i)}{D_i} \quad (3.1)$$

where $\hat{p}_{xi} = \hat{l}_{x+i} / \hat{l}_x$ is the observed probability of surviving from age x to age $x+i$ and D_i the number of deaths at age i (Chiang, 1968). As the number of deaths D_i increases (that is, when exposures increase), the sampling variance becomes increasingly small, and eventually approaches zero.

If the temporal unfolding of life expectancy at birth is viewed as a stochastic process $e_o(t)$ with mean $E(e_o(t)) = \mu(t)$ and variance $\text{Var}(e_o(t)) = E[e_o(t) - \mu(t)]^2 = \bullet^2(t)$,

then (3.1) estimates $\hat{\bullet}^2(\hat{e}_o) = E[\hat{e}_o(t) - e_o^r(t)]^2$ where $\hat{e}_o(t)$ is the observed life

expectancy and $e_o^r(t)$ hypothetically realized segment of $e_o(t)$ at time t . For a large

risk population, $\hat{e}_o(t) \approx e_o^r(t)$. On the other hand, if the risk population is small then the sampling error may be large (see e.g., Hartmann, 2003). In Sweden with a population of about 9 million, the sampling standard deviation of the life expectancy at birth

for either sex is about $\hat{\sigma}(\hat{e}_0) \approx 0.1$. This means that a 95 percent sample confidence interval for the observed life expectancy at birth, for either sex, is of magnitude $\hat{e}_0 \pm 0.2$ years. This suggests that when simulating a mortality experience in the context of making a stochastic population projection, two sources of variation may have to be taken into consideration; on the one hand, the sampling variance (especially if the risk population is very small) and, on the other, the variance of the underlying stochastic process. However, as already noted, when the risk population is large then the variation due to sampling is negligible so that only the process variance $\text{Var}(e_0(t)) = \sigma^2(t)$ has to be taken into account. We make this assumption in the sequel.

In the context of stochastic projections, we adopt the view that constant mortality prevails when the underlying survival process is a stationary stochastic process. This, it will be realized, is not the same as if the life expectancy (or some other mortality characteristic) were time invariant (the graphical representation of which would be a straight line parallel with the time axis). Instead, it involves that e.g., the observed life expectancy is a product of a realization of a stationary process with mean value $E(e_0(t)) = \mu$ and variance $\text{Var}(e_0(t)) = \sigma^2$. The assumption of normality paves the way for estimating μ and σ^2 from time averages (second order ergodicity).

We refer to σ^2 as the natural variation in life expectancy because this is the variability imposed by nature (including e.g., social and economic constraints) during a time stretch for which it is reasonable to assume that the life expectancy remains a stationary process. If, moreover, it is assumed that the variation in mortality, for practical purposes, is independent of the level of mortality (as well as of population size) then the $\hat{\sigma}_e$ in table 3.1 references natural variability in survivorship for the periods shown.

4.0 MORTALITY AND ITS EFFECTS UPON THE POPULATION

4.1 An application to the Swedish 1980 population

To further the discussion, consider the estimated regression models in table 3.1 for Swedish males and females, 1980-2002. For easy reference, these are given in table 4.1 below. The error terms in the models are assumed independent and normally distributed with zero mean and with the standard deviations given in table 4.1.

Table 4.1 Regression models for Swedish male and female mortality, 1980-2002

Sex	Regression model	$\hat{\sigma}_e$
Males	$\hat{e}(t) = -0.0298t + 0.0162 + e(t)$	0.023
Females	$\hat{e}(t) = -0.0219t - 0.0305 + e(t)$	0.025

Fig. 4.1 shows five hundred repetitions of projecting the Swedish 1980 population thirty years into the future with the assumption that $TFR = 1.8$, zero net-migration, and survivorship provided by the regression models in table 4.1 (five hundred projected population sizes for the year 2010). The regression representations in table 4.1 reference the standard survival functions for males and females in 1980, which are used to generate survival during the projection period. Fig. 4.1, of course, gives a hypothetical situation and is merely intended to portray the variability in population size as a function of stochastically time varying mortality.

Fig. 4.1. Five hundred repetitions of projecting the Swedish 1980 population thirty years into the future. Graph shows the population in the year 2010.

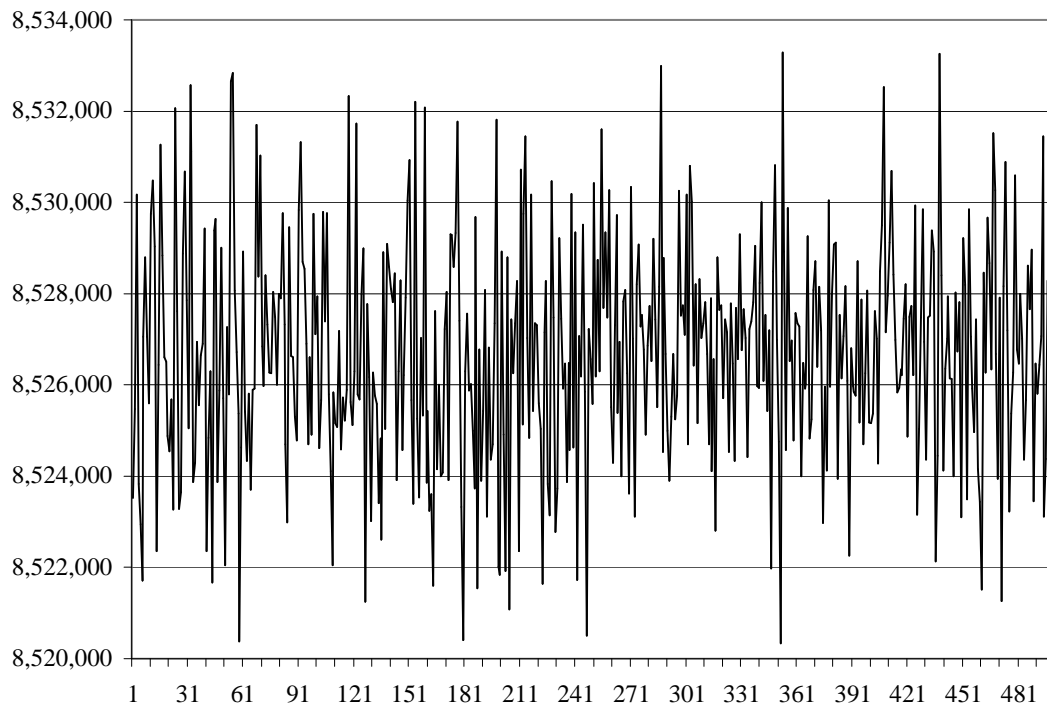


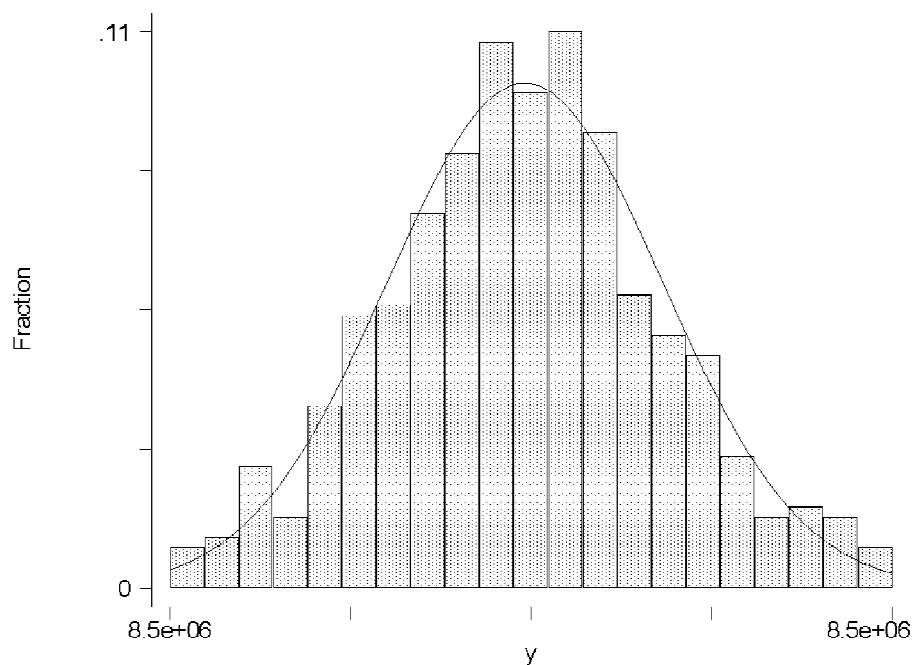
Table 4.2 shows the characteristics of the five hundred repetitions. The mean of the five hundred populations in year 2010 is 8,526,698, the minimum 8,520,341 and the maximum 8,533,276. The range is only 12,935 (table 4.2). This means that the random variation in the estimated models in table 4.1 only leads to marginal deviations, in terms of the total population, relative to a purely deterministic linear increase in life expectancy. While this also is what we would expect, nevertheless the experiment brings out the magnitude of variation due to stochastic effects in survival; it can be shown that the realization in fig. 4.1 is white noise.

As might be expected from table 4.2, which shows that the skewness and the kurtosis for the five hundred observations are close to nil, the associated frequency distribution is close to being a normal curve (fig. 4.2). The frequency distribution is not significantly different from a normal distribution with the same mean and variance as given by table 4.2 ($p > 0.9$).

Table 4.2. Characteristics for 500 repetitions of projecting the Swedish 1980 population thirty years into the future.

Mean	8,526,698
Standard deviation	2,463
Kurtosis	-0.07
Skewness	0.05
Range	12,935
Minimum	8,520,341
Maximum	8,533,276

Fig. 4.2. Frequency distribution for five hundred repetitions of projecting the Swedish 1980 population thirty years into the future.



The age distributions in the five hundred repetitions are so similar that, for practical purposes, they can be viewed as constant.

Fig. 4.3 shows the life expectancies corresponding to realizations of the regression models for males and females in table 4.1 and fig. 4.4 life expectancies corresponding to realizations of stationary processes with means $e_0 = 72.8$ and $e_0 = 78.8$ for males and females, respectively (table 4.3). For males the model is $\hat{e}(t) = 0.0162 + e(t)$ where $e(t)$ are independent normally distributed random numbers with standard deviation $\hat{\sigma}_e = 0.03$. The standard deviation for males and females are $\sigma(e_0) = 0.22$ and $\sigma(e_0) = 0.23$, respectively (table 4.3). These standard deviations are about twice those due to sampling in current life tables for Sweden.

Fig. 4.3. Simulated life expectancies for males and females, for one of the five hundred repetitions, during the 30 years of projection

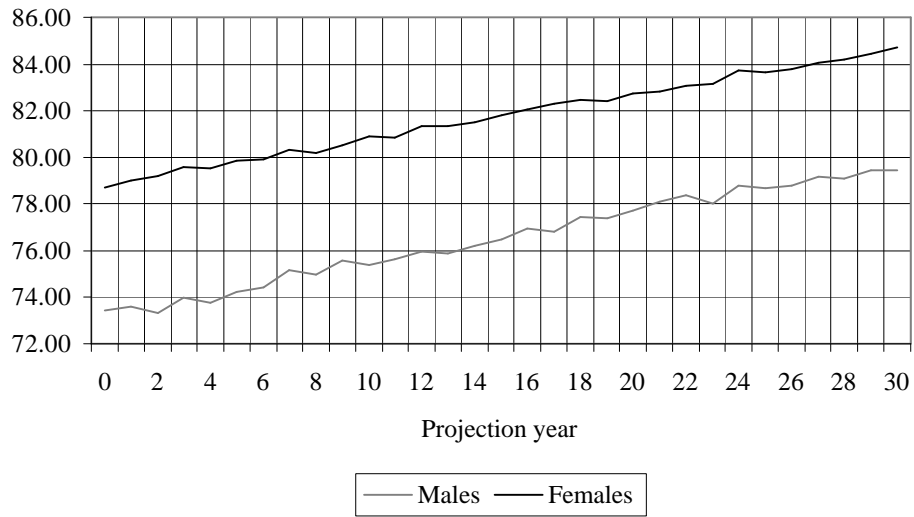


Fig. 4.4. Constant life expectancies portrayed as stationary normal processes

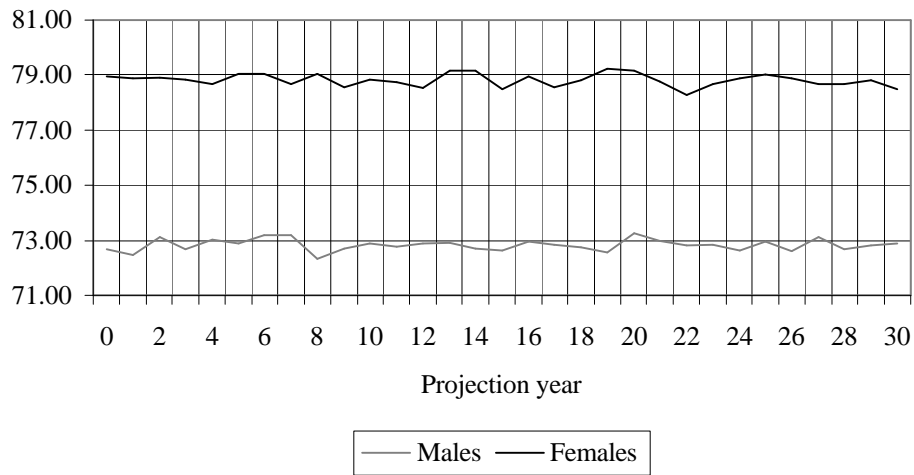


Table 4.3. Mean and standard deviation for stationary process of life expectancy

Characteristic	Males	Females
Mean	72.8	78.8
Standard deviation	0.22	0.23

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